On the computation of graded components of Laurent polynomial rings

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Abstract

In this paper, we present several algorithms for dealing with graded components of Laurent polynomial rings. To be more precise, let S be the Laurent polynomial ring $k[x_1,\ldots,x_r,x_{r+1}^{\pm 1},\ldots,x_n^{\pm 1}]$, k algebraicaly closed field of characteristic 0. We define the multigrading of S by an arbitrary finitely generated abelian group A. We construct a set of fans compatible with the multigrading and use this fans to compute the graded components of S using polytopes. We give an algorithm to check whether the graded components of S are finite dimensional. Regardless of the dimension, we determine a finite set of generators of each graded component as a module over the component of homogeneous polynomials of degree 0.

1 Introduction

The formal construction of the homogeneous coordinate ring of a toric variety, that I.M. Musson in [8], D.A. Cox in [3] and others discovered in the early 1990s, takes a fan Δ and creates a torus action on an open subset of an affine space whose quotient is the toric variety of Δ. We reverse this process in [12]. To be more specific, let k be an algebraically closed field of characteristic 0. We take a torus action on the affine space $X = k^r \times (k^\times)^s$ and create various fans whose toric varieties are the quotients under the action of the torus of an open subset of X. This "reverse engineering" has appeared in other places, see for instance Chapter 10 of Miller and Sturmfels book [9] where they take an algebraic torus times a finite abelian group acting on k^n.

In this paper, we extend the construction of the fan associated to the action given in [12] to cover also Miller and Sturmfels situation but we focus on making the process computational. Let H be an algebraic torus times a finite abelian group,

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we give an algorithm to obtain a set of fans associated to the diagonal action of H on X. Using these fans, we will be able to give a computational description of the graded components of Laurent polynomial rings.

Let S be the Laurent polynomial ring $k[x_1, \ldots, x_r, x_{r+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$, with $n = r+s$. The action of H on X extends to an action of H on S. As in the polynomial ring case, this action determines a multigrading of S by the finitely generated abelian group $A = Hom(H, k^{\times})$. Multigradings of polynomial rings are treated in detail in [9], Chapter 8. In section 4, we extend the definition of multigrading to Laurent polynomial rings and in section 5, we provide an algorithm to obtain a set of fans compatible with the multigrading of S by A. This fans allow us to see graded components of S as polyhedra. Each graded component S_a, $a \in A$ is a module over the ring of differential invariants $\mathcal{D} = k[x_1, \ldots, x_r, x_{r+1}^{\pm 1}, \ldots, x_n^{\pm 1}, \partial_1, \ldots, \partial_n]^H$ where $\partial_i = \partial/\partial x_i$, $i = 1, \ldots, n$. In [10], we studied graded components of S from this point of view . The connection between group actions and finite fans allows a computational description of this family of $\mathcal{D}$-modules that we present in this paper.

When computing the graded components of S, it is important to decide whether they are finite dimensional as k vector spaces or not. Let S_0 be the k vector space of homogeneous polynomials of degree 0. As in the polynomial ring case, either all the graded components are finite or infinite dimensional as we conclude in the last section of the paper. In section 7, we give an algorithm to check whether S_0 is finite dimensional.

Given a finite fan Δ, a shift with nonempty intersection of the dual cones of the cones of Δ gives a polyhedron. Furthermore, such a polyhedron is a polytope if and only if the fan is not contained in a half-space, as we prove in section 3. In section 8, given a fan compatible with the multigrading of S by A, we use polyhedra obtained in this manner to give a set of generators for each S_a as a module over S_0. If the graded components of S are finite dimensional we compute a k-basis of S_a using the lattice points of a polytope. We provide a set of linear inequalities determining this polytopes so that the number of lattice points can be counted with LattE, [4]. When S_0 is infinite dimensional, we determine a finite set of generators $f_1, \ldots, f_h$, corresponding to the Hilbert basis of a cone, such that $S_0 = k[f_1, \ldots, f_h]$. To determine the infinite dimensional S_a, we compute a finite set of generators of S_a as an S_0-module corresponding to the lattice points of a polytope.

We begin our paper describing in section 2 the group H and the diagonal action of H on X. Throughout this paper, a particular action will be specified by a matrix L whose columns are the weights of the action. In some special cases, we can replace the matrix L by another one with the special form given in section 6 which gives the same action. Most of the algorithms given in this paper will take the matrix L or its special form as an input.

2 Description of the group and the action

We consider a diagonal action of H on $X = k^r \times (k^{\times})^s \subseteq k^n$ with $n = r + s$. This is an action that extends to a diagonal action on k^n. Such an action is given by an embedding of H into the group T of diagonal matrices in $GL_n(\mathbb{Z})$. Details about this

action in the algebraic torus case are given in [10], §2.1. The construction developed there applies to the more general case treated here.

Identify H with $G \times \mathbb{F}$ where $G = (k^\times)^p$ and $\mathbb{F}$ is the product of cyclic groups of orders $d_1, \ldots, d_t$. We can identify the group of characters of H, $A = Hom(H, k^\times)$ with

$$A = \mathbb{Z}^p \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/d_t\mathbb{Z}.$$

We think of A as a space of column vectors with integer entries.

There exist $\eta_1, \ldots, \eta_n \in A$ such that H *acts on* X *with weights* $\eta_1, \ldots, \eta_n$. Let $m = p+t$. We denote by L the $m \times n$ matrix with i-th column vector η_i, $i = 1, \ldots, n$. We say that H *acts on* X *by the matrix* L. We may and we will assume that H acts faithfully on X. Therefore L has rank m.

Let us also denote by D the diagonal matrix with entries $d_1, \ldots, d_t$. This $t \times t$ matrix will be used in several computations throughout this paper.

3 Finite fans

In this section we state some results about finite fans that will be needed in the rest of the paper. As far as possible we follow the notation of [5] , Chapter 1.

Let $N = \mathbb{Z}^l$ be the l-dimensional lattice. Let Δ be a fan in N, which is a collection of strongly convex rational polyhedral cones in $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R} \equiv \mathbb{R}^l$. Through this paper, cones will be strongly convex, rational and polyhedral except in a few occasions where we will say convex cone to mean convex rational polyhedral cone.

Let $M = Hom_\mathbb{Z}(N, \mathbb{Z})$ and $\langle \_ , \_ \rangle : M \times N \to \mathbb{Z}$ the natural bilinear pairing. For each $\sigma \in \Delta$, let

$$\Lambda_\sigma = M \cap \sigma^\vee = \{u \in M | \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}. \tag{1}$$

We review some terminology introduced in [12], §1.2. Denote by $\Delta(1)$ the set of cones of Δ with dimension one. Given $v \in N$, let $\tau_v = \mathbb{R}_+ v$ be the ray generated by $v \in N$. Given $\sigma \in \Delta$ we define $[\sigma] = \{i \in \{1, \ldots, r\} \mid \tau_{v_i} \text{ is a face of } \sigma\}$. Given $u \in M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R} \equiv \mathbb{R}^l$, a subset of the form $H_u = \{v \in N_\mathbb{R} \mid \langle u, v\rangle \geq 0\}$ with $u \neq 0$ is called a half-space in $N_\mathbb{R}$. We say that the fan Δ is *contained in a half-space* if we can find $0 \neq u \in M_\mathbb{R}$ such that $\sigma \subseteq H_u$ for all $\sigma \in \Delta$.

Given a fan Δ, let $\{v_1, \ldots, v_r\}$ be a set of vectors in N generating the one dimensional cones of Δ. Fix $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{N}^r \times \mathbb{Z}^s$ and define the sets

$$\gamma_{\sigma,\varphi} = \{u \in M_\mathbb{R} \mid \langle u, v_i\rangle \geq -\varphi_i \text{ for all } i \in [\sigma]\} \tag{2}$$

for each $\sigma \in \Delta$. Consider the polyhedron

$$\mathcal{P}_\varphi = \cap_{\sigma \in \Delta} \gamma_{\sigma,\varphi} = \{u \in M_\mathbb{R} \mid \langle u, v_i\rangle \geq -\varphi_i \text{ for all } i = 1, \ldots, r\}. \tag{3}$$

The purpose of the following two lemmas is to prove that $\mathcal{P}_\varphi$ is bounded if and only if the fan Δ is not contained in a half-space.

Lemma 3.1. *Fix an integer $a \geq 0$ and $v \in \mathbb{Z}^l$ and consider the set $\Omega = \{u \in \mathbb{R}^l \mid \langle u, v\rangle \geq -a\}$. Given $u \in S = \{u \in \mathbb{R}^l | |u| = 1\}$, let $l_u = sup\{\mu \in \mathbb{R}^+ | \mu u \in \Omega\}$. Then the map $f : S \to \mathbb{R}^l$ defined by $f(u) = l_u$ is continuous in its domain.*

Proof. It can be easily seen that if $l_u < \infty$, then $[0, l_u] = \{\mu \in \mathbb{R}^+ | \mu u \in \Omega\}$. In fact, $\langle l_u u, v\rangle = -a$, i.e. if $v = (v_1, \ldots, v_l)$ then $l_u u$ belongs to the hyperplane of $\mathbb{R}^l$ given by the equation $v_1x_1 + \ldots + v_lx_l = -a$. On the other hand $l_u u$ also belongs to the line in $\mathbb{R}^l$ through the origin in the direction of $u = (u_1, \ldots, u_l)$. Therefore $l_u u$ is solution of the matrix equation $M_u X = B$, where

$$M_u = \begin{bmatrix} v_1 & v_2 & \cdots & v_l \\ u_2 - u_1 & & & \\ & u_3 - u_2 & & \\ & & \ddots & \\ & & & u_l - u_{l-1} \end{bmatrix}, \qquad B = \begin{bmatrix} -a \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and X is the column vector of unknowns.

Let M_u^i be the matrix obtained exchanging the i-th column of M_u by B. Using Cramer's Rule, and the fact $l_u = |l_u u|$, we obtain the following expression for $f(u)$

$$f(u) = l_u = \left(\sum_{i=1}^{r} \left(\frac{detM_u^i}{detM_u}\right)^2\right)^{\frac{1}{2}}$$

This proves the result. □

Lemma 3.2. *The set $\mathcal{P}_\varphi$ is bounded if and only if there does not exist $u \in M_\mathbb{R}$, $u \neq 0$ such that $\mathbb{R}^+ u \subset \mathcal{P}_\varphi$.*

Proof. Obviously if $\mathcal{P}_\varphi$ is bounded the conclusion is clear. Let us prove the other direction. Consider the sets

$$\Omega_i = \{u \in M_\mathbb{R} \mid \langle u, v_i\rangle \geq -\varphi_i\}$$

for all $i \in \cup_{\sigma \in \Delta} [\sigma] = \{1, \ldots, r\}$. For each $\sigma \in \Delta$ we have $\gamma_{\sigma,\varphi} = \cap_{i \in [\sigma]} \Omega_i$. Then $\mathcal{P}_\varphi = \cap \Omega_i$ where the intersection is taken over all the $i \in \{1, \ldots, r\}$. Let

$$l_u^i = sup\{\mu \in \mathbb{R}^+ \mid \mu u \in \Omega_i\} \tag{4}$$

for $u \in \mathbb{R}^l$ and $i \in \{1, \ldots, r\}$. Then for each $i \in \{1, \ldots, r\}$ we define maps $f_i : S \longrightarrow \mathbb{R}^+$ where $f_i(u) = l_u^i$ for every $u \in S$. Each f_i is continuous in its domain by Lemma 3.1. Now we define the map $F : S \longrightarrow \mathbb{R}^+$ where

$$F(u) = \inf\{l_u^i \mid i \in \{1, \ldots, r\}\} \tag{5}$$

for all $u \in S$. By [11], Chapter 2, Section 18, Exercise 8, F is continuous in its domain. By hypothesis, there is no $u \in M_\mathbb{R}$, $u \neq 0$ such that $\mathbb{R}^+ u \subseteq \mathcal{P}_\varphi$. Therefore the domain of F is S. By [2], Chp. II 5.8, $F(S^{r-1})$ is compact in $\mathbb{R}^+$ and by the Heine-Borel theorem it is also bounded. Therefore we can find a positive integer N such that $F(u) \leq N$ for all $u \in S$. If $y \in \mathcal{P}_\varphi$, then $y = \mu u$ for some $u \in S$ and by (5) and (4), $\mu \leq F(u) \leq N$. Therefore $\mathcal{P}_\varphi$ in included in the sphere of radius N. □

Proposition 3.3. *The set $\mathcal{P}_\varphi$ is a polytope if and only if the fan Δ is not contained in a half-space.*

Proof. Suppose there exists $u \in M_{\mathbb{R}}$, $u \neq 0$ such that the fan Δ is contained in the half-space H_u. Then given $i \in \{1, \ldots, r\}$ and for any $\mu \in \mathbb{R}^+$ it holds that $\langle \mu u, v_i \rangle \geq 0 \geq -\varphi_i$. Thus $\mathbb{R}^+ u \subseteq \mathcal{P}_\varphi$ and hence $\mathcal{P}_\varphi$ is not bounded.

Suppose now that $\mathcal{P}_\varphi$ is not bounded. Then by Proposition 3.2 there exists $u \in M_{\mathbb{R}}$, $u \neq 0$ such that $\mathbb{R}^+ u \subseteq \mathcal{P}_\varphi$. By (2), we have $\mu \langle u, v_i \rangle \geq -\varphi_i$ for every $i \in \{1, \ldots, r\}$ and every $\mu \in \mathbb{R}^+$. Then $\langle u, v_i \rangle \geq 0$ for all $i \in \{1, \ldots, r\}$. Therefore the fan Δ is contained in the half-space H_u. □

Remark 3.4. *A fan Δ is not contained in a half-space if and only if the intersection of the dual cones is zero, $\cap_{\sigma \in \Delta} \sigma^\vee = 0$. Moreover, given a set of generators $\{v_1, \ldots, v_r\}$ of cones in $\Delta(1)$ and $0 \neq u \in M_{\mathbb{R}}$, the following statements are equivalent:*

1. *Δ is contained in the half-space H_u.*
2. *The vector u is a solution in $\mathbb{R}^l$ of the system of linear inequalities $\{\langle x, v_i \rangle \geq 0, i = 1, \ldots r\}$.*
3. *The set $\mathbb{R}_{\geq 0}\{v_1, \ldots, v_r\}$ is a convex cone contained in the half-space H_u.*

There are well known algorithms to find a nonzero solution of a system of linear inequalities $\{\langle x, v_i \rangle \geq 0, i = 1, \ldots, r\}$. Because we need to use it in section 7, we give next an algorithm to find such a solution in the language of half-spaces. The algorithm is based on the following lemma.

Lemma 3.5. *Let σ be a cone in $N_{\mathbb{R}}$ and let $v \in N_{\mathbb{R}}$. Then $\mathcal{C} = \mathbb{R}_{\geq 0}(\sigma \cup \{v\})$ is a convex cone if and only if $\mathcal{C} \subseteq H_u$ for some $u \in M_{\mathbb{R}}$ such that $\langle u, v \rangle = 0$.*

Proof. Let us suppose that $\{v_1, \ldots, v_r\}$ is a set of generators of σ. Then $\{v_1, \ldots, v_r, v\}$ is a set of generators of $\mathcal{C}$. If $\mathcal{C}$ is a cone, by [13], Theorem 7.1 it is the intersection of the half-spaces H_u, $u \in M_{\mathbb{R}}$ containing $\{v_1, \ldots, v_r, v\}$ and such that $\{v \in N_{\mathbb{R}} \mid \langle u, v \rangle = 0\}$ is spanned by $l - 1$ linearly independent vectors from $\{v_1, \ldots, v_r, v\}$. Let H_{u_0} be one the this half-spaces such that $\langle u_0, v \rangle = 0$. This proves the result. □

Algorithm 3.6. HSP1*($\{w_1, \ldots, w_t\}$,σ , u)* Given *a convex cone σ contained in the half-space H_u with $0 \neq u \in M_{\mathbb{R}}$ and a set of vectors $\{w_1, \ldots, w_t\}$ in $N_{\mathbb{R}}$, the algorithm* decides *whether $\mathcal{C} = \mathbb{R}_{\geq 0}(\sigma \cup \{w_1, \ldots, w_t\})$ is contained in a half-space and if the answer is affirmative it* returns *a nonzero vector u_0 such that $\mathcal{C}$ is contained in H_{u_0}.*

1. *$i := 1$, $\mathcal{C} := \sigma$.*
2. *While $i \leq t$ do*

    2.1 If $w_i \in \mathcal{C}$ then i:=i+1

2.2 else take $u \in M_{\mathbb{R}}$ such that $\langle u, w_i \rangle = 0$. If $\mathcal{C} \subseteq H_{\pm u}$ then $\mathcal{C} := \mathbb{R}_{\geq 0}(\mathcal{C} \cup \{w_i\})$, $i := i+1$, $u_0 := \pm u$ else return "Not contained in a half-space".

3. Return u_0 and "Contained in a half-space".

Let $\mathcal{V} = \{v_1, \ldots, v_r\}$ be a set of generators of a fan Δ in N. If $\mathcal{V}$ has rank less than l then Δ is contained in a half-space. Otherwise $\mathcal{V}$ contains a subset $\mathcal{W}$ with l linearly independent vectors. By [13], Theorem 7.1, $\mathbb{R}_{\geq 0}\mathcal{W}$ is a cone contained in the half-space H_u for some $u \in M_{\mathbb{R}}$ such that $\mathcal{W} \subseteq H_u$ and $\{v \in N_{\mathbb{R}} \mid \langle u, v \rangle = 0\}$ is spanned by $l-1$ vectors of $\mathcal{W}$. Algorithm 3.6 applied to $(\mathcal{V} \setminus \mathcal{W}, \mathbb{R}_{\geq 0}\mathcal{W}, u)$ decides whether Δ is contained in a half-space.

4 Multigradings

The Laurent polynomial ring S is the ring of regular functions on X, $\mathcal{O}(X)$. We consider the action of H on $\mathcal{O}(T)$ (or $\mathcal{O}(X)$) given by right translation, see [10], (12). This convention implies that x_i has weight η_i.

We say that a Laurent polynomial ring S is *multigraded* by A when it has been endowed with a degree map $deg : \mathbb{Z}^n \to A$. Multigradings of polynomial rings are defined in [9], §8.1 in the same manner.

Using the matrix L that gives the action of H on X, we define a degree map as follows. The restriction map $\mathbb{X}(T) = Hom(T, k^{\times}) \longrightarrow \mathbb{X}(H)$ is given by left multiplication by L. If we identify $\mathbb{X}(T)$ with $\mathbb{Z}^n$ we have an exact sequence (see lemma 4.1)

$$0 \longleftarrow A \longleftarrow \mathbb{Z}^n \longleftarrow \mathfrak{K} \longleftarrow 0 \tag{6}$$

which determines a multigrading of S by A. Given $\lambda = (\lambda_1, \ldots, \lambda_n)$, we write $x^{\lambda} = x_1^{\lambda_1} \cdot \ldots \cdot x_n^{\lambda_n}$. The middle semigroup homomorphism $deg : \mathbb{Z}^n \to A$ gives a multigrading of S by A sending each monomial x^{λ} to its *degree* $deg(\lambda) = L\lambda$. We will say that the *multigrading of S by A is given by the matrix L*. Let $Q = deg(\mathbb{N}^r \times \mathbb{Z}^s)$ denote the semigroup of A generated by $deg(x_1), \ldots, deg(x_r), \pm deg(x_{r+1}), \ldots, \pm deg(x_n)$.

The next lemma proves that (6) is an exact sequence. Let $l = n - p$.

Lemma 4.1. *The degree map deg is surjective and there exists an $n \times l$ matrix K whose columns are a $\mathbb{Z}$-basis of $\mathfrak{K}$.*

Proof. By [1], Theorem 12.4.3, there exist matrices $U \in GL_m(\mathbb{Z})$ and $V \in GL_n(\mathbb{Z})$ such that

$$L' = ULV = \begin{bmatrix} f_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & f_2 & & 0 & & & \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & \ldots & & f_m & 0 & \ldots & 0 \end{bmatrix}, \tag{7}$$

with $f_i \neq 0$ for all $i = 1, \ldots, m$. Consider the $n \times l$ matrix

$$E = \begin{bmatrix} 0 & 0 \\ 0 & D \\ I_{l'} & 0 \end{bmatrix} \tag{8}$$

with D as in section 2, the integer $l' = l - t$ and $I_{l'}$ the $l' \times l'$ identity matrix.

The columns of E are a $\mathbb{Z}$-basis of the kernel of the group homomorphism $\mathbb{Z}^n \longrightarrow A$ given by right multiplication by L'. Let $K := VE$, then $LK = 0$. Therefore the columns of K are a $\mathbb{Z}$-basis of the kernel of L. We can write $\mathfrak{K}$ as $K\mathbb{Z}^l$. Then K is a presentation of $deg(\mathbb{Z}^n)$. By [1], Proposition 5.12 we also have a presentation $E = V^{-1}K$ of $deg(\mathbb{Z}^n)$. We conclude that $deg(\mathbb{Z}^n) = A$. $\square$

We are ready to introduce the graded components of S that we are computing in this paper. For $a \in A$, let S_a denote the vector space of homogeneous polynomials having degree a in the A-grading, that is

$$S_a = span\{x^\lambda \in S \mid L\lambda = a\}. \tag{9}$$

Therefore $S = \bigoplus_{a \in Q} S_a$ since S_a is empty if $a \notin Q$ and $S_a \cdot S_b = S_{a+b}$, $a, b \in Q$. Note that the subring S^H of S of invariants under the action of H equals the semigroup ring

$$S_0 = k[\mathfrak{K} \cap (\mathbb{N}^r \times \mathbb{Z}^s)]. \tag{10}$$

5 Construction of fans compatible with the multigrading

The concept of fan compatible with the multigrading of a polynomial ring was given in [9], section 10.3. Next, we extent the definition so that it applies to multigradings of Laurent polynomial rings. In this section, we give an algorithm to construct a fan compatible with a multigrading from the matrix L describing the action of H on X.

Applying the contravariant functor $Hom(\_, \mathbb{Z})$ to the sequence (6) we obtain the exact sequence

$$0 \longrightarrow Hom(A, \mathbb{Z}) \longrightarrow \mathbb{Z}^n \longrightarrow \mathfrak{K}^\vee \longrightarrow Ext^1(A, \mathbb{Z}) \longrightarrow 0. \tag{11}$$

By the previous lemma 4.1, $\mathfrak{K}$ is a free abelian group of dimension l. Then we identify $\mathfrak{K}$ and $\mathfrak{K}^\vee = Hom(\mathfrak{K}, \mathbb{Z})$ with $\mathbb{Z}^l$. We fix the columns of the matrix K given by lemma 4.1 as a $\mathbb{Z}$-basis of $\mathfrak{K}$.

The middle morphism $\varphi : \mathbb{Z}^n \longrightarrow \mathfrak{K}^\vee \equiv \mathbb{Z}^l$ of the sequence (11) is given by right multiplication by the matrix K. Let e_i be the i-th standard basis vector of $\mathbb{Z}^n$ and call $v_i = \varphi(e_i)$ the i-th row vector of K, $1 \le i \le n$. In particular, φ is onto when H is an algebraic torus, see [12], paragraph following Lemma 2.2. We call $\{v_1, \ldots, v_r\}$ a *set of vectors associated to the action* of H on X.

We call a fan Δ in N *compatible* with the multigrading of S by A if its set of one dimensional cones equals $\Delta(1) = \{\tau_{v_i} \mid i = 1, \ldots, r\}$. The cones of Δ are subcones of $\mathcal{C} = \mathbb{R}_{\ge 0}\{v_1, \ldots, v_r\}$.

The proof of lemma 4.1 provides an algorithm to compute a set of vectors associated to the action of H on X.

Algorithm 5.1. ASSOCIATED-VECTORS*(L,D)* Given *the $m \times n$ matrix L associated to the action of H on $X = k^r \times (k^\times)^s$ and the $t \times t$ matrix D given in section 2, the algorithm* returns *a set of vectors $\{v_1, \ldots, v_r\}$ associated to the action of H on X.*

1. *Compute the Smith normal form of L; that is, compute invertible matrices U and V such that $U \cdot L \cdot V$ is the concatenation of a diagonal matrix and a zero matrix.*

2. *Let $l' := n - m$, and*

$$E := \begin{bmatrix} 0 & 0 \\ 0 & D \\ I_{l'} & 0 \end{bmatrix}. \tag{12}$$

3. *$K := V \cdot E$.*

4. *Return the set $\{v_1, \ldots, v_r\}$ of the first r row vectors of K.*

6 Special form of the matrix L

For some of the constructions in the next sections we need the matrix L (introduced in section 2) to have the special form presented in the next lemma. Now suppose H acts on X via the matrix L and set

$$\Sigma_L = \{\lambda \in \mathbb{N}^r \times \mathbb{Z}^s \mid deg(\lambda) = 0_A\}, \tag{13}$$

where 0_A is the class of zero in A. We show that in some cases we can obtain a matrix L' equivalent to L with the special form and giving an action of H on X with the same set of invariants.

Given a matrix M we will write $subm(M, r_1 \ldots r_2, c_1 \ldots c_2)$ to denote the submatrix of M obtained selecting rows r_1 through r_2 and columns c_1 through c_2 of M.

Lemma 6.1. *If $\eta_{r+1}, \ldots, \eta_n$ are linearly independent, there exist matrices $\Gamma \in GL_m(\mathbb{Z})$, $\Delta \in GL_n(\mathbb{Z})$ such that*

1. *$L' = \Gamma \cdot L \cdot \Delta$ has the block matrix form*

$$\begin{bmatrix} L_1 & dI_p \\ L_3 & L_4 \end{bmatrix}, \tag{14}$$

where I_p is the $p \times p$ identity matrix and d is a nonzero positive integer.

2. *Σ_L is isomorphic to Σ'_L.*

Proof. 1. Since L has rank $m = p+t$ there exists a matrix $\Delta \in GL_n(\mathbb{Z})$ such that the last p columns of $L \cdot \Delta$ are linearly independent and such that $\Delta x \in \mathbb{N}^r \times \mathbb{Z}^s$ for any $x \in \mathbb{N}^r \times \mathbb{Z}^s$. To be more precise, if $p \leq s$ then Δ is the identity matrix since $\eta_{r+1}, \ldots, \eta_n$ are linearly independent whereas if $p > s$ then Δ is a permutation of the first r columns of L.

Furthermore, we can find a matrix $\Gamma_1 \in GL_m(Z)$ permuting the rows of $L \cdot \Delta$ such that $L_2 = subm(\Gamma_1 \cdot L \cdot \Delta, 1 \ldots p, l+1 \ldots n)$ has rank p. Let $d = det(L_2)$. Then $\Gamma_1 = dL_2^{-1}$ is matrix in $GL_p(\mathbb{Z})$. Let

$$\Gamma = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & D \end{bmatrix}$$

then $\Gamma \cdot L \cdot \Delta$ has the desired form (14).

2. Since Δ permutes only the first r columns of L and $\Gamma 0_A = 0_A$ we can define the following isomorphism
$$\Sigma_L \longrightarrow \Sigma_{L'} \tag{15}$$
given by $x \mapsto \Delta^{-1}x$.

$\square$

From the proof of statement 1. in the previous lemma we derive the following algorithm.

Algorithm 6.2. SPECIAL-MATRIX*(L,D)* Given *the $m \times n$ matrix L associated to the action of H on $X = k^r \times (k^\times)^s$, where the columns $\eta_{r+1}, \ldots, \eta_n$ are linearly independent, and the $t \times t$ matrix D given in section 2, the algorithm* returns *an $m \times n$ matrix L' with the special from (14).*

1. *$p := m - t$, $l := n - p$.*
2. *If $p > s$ then permute the first r columns of L to obtain L_c so that $rank(subm(L_c, 1 \ldots n, l+1 \ldots r)) = p - s$.*
3. *$L := L_c$. Permute the rows of L to obtain L_r so that $rank(subm(L_r, 1 \ldots p, l + 1 \cdots n)) = p$.*
4. *$L := L_r$. $L_2 := subm(L, 1 \ldots p, l+1 \ldots n)$, $d := det(L_2)$, $\Gamma_2 := dL_2^{-1}$.*
5. *$\Gamma := \begin{bmatrix} \Gamma_2 & 0 \\ 0 & D \end{bmatrix}$.*
6. *Return $L' := \Gamma \cdot L$.*

7 Determining the special case $S_0 = k$

To describe the graded components of S we need to decide in the first place whether they are finite or infinite dimensional. For that matter, it is enough to decide whether S_0 is finite dimensional as we will see in section 8. In this section, we give an algorithm to decide whether $S_0 = k$ or equivalently whether S_0 is finite dimensional.

We define next an isomorphism allowing us to go from points in the lattice $M \equiv \mathbb{Z}^l$ to points in the lattice $\mathfrak{K} \subset \mathbb{Z}^n$ and we use it to prove the next proposition. There is a natural bilinear pairing $(\ ,\ ) : \mathbb{X}(T) \times \mathbb{Y}(T) \longrightarrow \mathbb{Z}$ defined by the requirement that $(a \circ b)(\lambda) = \lambda^{(a,b)}$ for all $a \in \mathbb{X}(T)$, $b \in \mathbb{Y}(T) = Hom(k^\times, T)$ and $\lambda \in k^\times$. We have an isomorphism $\omega : M \to \mathfrak{K}$ given by
$$\langle u, \varphi(b) \rangle = (\omega(u), b) \tag{16}$$
for all $u \in M$, $b \in \mathfrak{K}$.

Theorem 7.1. *The following statements are equivalent.*

1. *The only polynomials of degree 0 are the constants; i.e.* $S_0 = k$.

2. *The* k*-vector space* S_0 *is finite dimensional.*

3. *Any fan* Δ *compatible with the multigrading of* S *given by* A *is not contained in a half-space.*

Proof. The proof is analogous to [12], Lemma 4.2. Let Δ be a fan compatible with the multigrading of S given by A. Let

$$\phi_\sigma = \{\lambda \in \mathfrak{K} \mid (\lambda, e_i) \geq 0 \text{ for all } i \in [\sigma]\}. \tag{17}$$

Then by (10), $S_0 = k[\cap_{\sigma\in\Delta}\phi_\sigma]$. Therefore S_0 is finite dimensional if and only if $\cap_{\sigma\in\Delta}\phi_\sigma$ is a finite set. Furthermore $\omega(\cap_{\sigma\in\Delta}\Lambda_\sigma) = \cap_{\sigma\in\Delta}\phi_\sigma$ and $\cap_{\sigma\in\Delta}\Lambda_\sigma$ is a finite set if and only if $\cap_{\sigma\in\Delta}\sigma^\vee$ is bounded. This happens if and only if $\cap_{\sigma\in\Delta}\sigma^\vee = 0$ which is equivalent to statements 1. and 3.. □

An A-grading of S verifying any of the equivalent conditions given in the previous proposition is called *positive*.

Proposition 7.2. *The following are necessary conditions for the grading of* S *by* A *to be positive.*

1. $p > s$.

2. $\eta_{r+1}, \ldots, \eta_n$ *are linearly independent.*

Proof.

1. Let $\{v_1, \ldots, v_r\}$ be a set of vectors associated to the action of H on X. If $p \leq s$ or equivalently $l \geq r$ then $\mathcal{C} = \mathbb{R}_{\geq 0}\{v_1, \ldots, v_r\}$ is a cone and equivalently Δ is contained in a half-space. By theorem 7.1 the result follows.

2. Let $\hat{G}$ be the m dimensional algebraic torus, then $H \subset \hat{G}$. Consider the action of $\hat{G}$ on X given by the matrix L, then $S^{\hat{G}} \subseteq S^H$. If $\eta_{r+1}, \ldots, \eta_n$ are linearly dependent, [10], Lemma 4.1(1) implies that $S^{\hat{G}}$ is not equal to k and therefore S_0 is not finite dimensional.

□

Let us suppose that $\eta_{r+1}, \ldots, \eta_n$ are linearly independent. Then we can assume that L has the special form (14). Let $\mathcal{V} = \{v_1, \ldots, v_r\}$ be the output of algorithm 5.1 applied to L_1 and D, and let Δ be a fan compatible with the multigrading of S by A given by L.

For $1 \leq i \leq l$, let ρ_i be the restriction of η_i to the subtorus G of H. These characters can be thought of as the columns of L_1.

Lemma 7.3. *If* $S_0 = k$ *then* $\rho_i \neq 0$ *for all* $i = 1, \ldots, l$.

Proof. By [12], Lemma 4.2 if $\rho_i = 0$ for some $i = 1, \ldots, l$ then Δ is contained in a half-space and by proposition 7.1 the result follows. □

The converse of the previous lemma does not hold but we can modify the matrix L to get an action of H on X whose only invariants are the constant polynomials whenever $\rho_i \neq 0$ for all $i = 1, \ldots, l$. Given a set $I \subseteq \{1, \ldots, r\}$ define

$$v_i^I = \begin{cases} -v_i \text{ if } i \in I \\ v_i \text{ if } i \notin I \end{cases}, i = 1, \ldots, r. \tag{18}$$

Let Δ_I be a fan in N with $\Delta_I(1) = \{\tau_{v_i^I} | i = 1, \ldots, r\}$. For $1 \leq i \leq n$, set

$$\varsigma_i = \begin{cases} -\eta_i \text{ if } i \in I \\ \eta_i \text{ if } i \notin I \end{cases}. \tag{19}$$

Let L_I be the matrix with columns $\varsigma_1, \ldots, \varsigma_n$. Then H_I denotes the group H acting on X by the matrix L_I. Then the fan Δ_I is compatible with the multigrading of S by A given by L_I.

We explain next how to obtain a set I so that $S^{H_I} = k$ whenever $\rho_i \neq 0$ for all $i = 1, \ldots, l$. Let us call $b_{i,j}$ the entries of L_1, $i = 1, \ldots, p$, $j = 1, \ldots, l$.

Lemma 7.4. *When the matrix L is of the special kind (14), then $v_1, \ldots, v_l$ are linearly independent.*

Proof. By Lemma 4.1, $LK = 0$. Then for $j = l+1, \ldots, n$

$$dv_j = -\sum_{i=1}^{l} b_{j-l,i} v_i. \tag{20}$$

Thus $v_{l+1}, \ldots, v_n$ belong to the $\mathbb{R}$-span of $v_1, \ldots, v_l$. By lemma 4.1 K has rank l and the result follows. □

By Lemma 7.4, $\mathcal{B} = \{v_1, \ldots, v_l\}$ is a basis of $N_{\mathbb{R}}$ and respect to $\mathcal{B}$ the vectors $v_{l+1}, \ldots, v_n$ have coordinates

$$v_j = -\frac{1}{d}(b_{j-l,1}, \ldots, b_{j-l,l}), \qquad j = l+1, \ldots, n. \tag{21}$$

Let $v_1^*, \ldots, v_l^*$ be the dual basis of $\mathcal{B}$. Then

$$\langle v_i^*, v_j \rangle = \frac{-1}{d} b_{j-l,i} \tag{22}$$

for $i = 1, \ldots, l$, $j = l+1, \ldots, r$.

Let us suppose that $p > s$ or equivalently $l < r$. Given $j \in \{l+1, \ldots, r\}$, let

$$I_j^+ = \{i \in \{1, \ldots, l\} \mid \frac{-1}{d} b_{j-l,i} > 0\}, \tag{23}$$

$$I_j^- = \{i \in \{1, \ldots, l\} \mid \frac{-1}{d} b_{j-l,i} < 0\}. \tag{24}$$

Lemma 7.5. *If $\rho_i \neq 0$ for all $i = 1, \ldots, l$ then there exists $J \in \{l+1, \ldots, r\}$ such that*

$$\cup_{j=1}^{J}(I_j^+ \cup I_j^-) = \{1, \ldots, l\}. \tag{25}$$

Proof. Otherwise there exists $i \in \{1, \ldots, l\}$ such that

$$\frac{-1}{d} b_{j-l,i} = 0,\ j = l+1, \ldots, r,$$

then $\rho_i = 0$. □

This lemma ensures that the next algorithm terminates.

Algorithm 7.6. POSITIVITY-SET*(L_1, d)* Given *the matrix L_1, where the columns $\rho_i \neq 0$ for all $i = 1, \ldots, l$, and the positive integer d in the special form of the matrix L giving the action of H on X, the algorithm* returns *a chain of subsets $\mathcal{I}_{l+1} \subseteq \ldots \subseteq \mathcal{I}_J$ of $\{1, \ldots, l\}$ such that the grading of S by A given by $L_{\mathcal{I}_J}$ is positive.*

1. *$J := l + 1$.*
2. *Compute I_J^- and I_J^+ using* (23) *and* (24)*.*
3. *$\mathcal{I}_J := I_J^+$.*
4. *While $\cup_{j=l+1}^{J}(I_j^- \cup I_j^+) \neq \{1, \ldots, l\}$ then*
   - *4.1 $I_J^0 := \{1, \ldots, l\} \setminus I_J$.*
   - *4.2 Compute I_{J+1}^- and I_{J+1}^+.*
   - *4.3 $\mathcal{I}_{J+1} := \mathcal{I}_J \cup ((\cap_{j=l+1}^{J} I_j^0) \cap I_{J+1}^+)$.*
   - *4.4 $J := J + 1$.*
5. *Return $\mathcal{I}_{l+1}, \ldots, \mathcal{I}_J$.*

The next proposition shows that the grading of S by A given by $L_{\mathcal{I}_J}$, with $\mathcal{I}_J$ as in the output of algorithm 7.6, is positive.

Proposition 7.7. *Let us suppose that $\rho_i \neq 0$ for all $i = 1 \ldots l$ and let $\mathcal{I}_J$ be the subset of $\{1, \ldots, l\}$ obtained by* POSITIVITY-SET*(L_1,d), then the grading of S by A given by $L_{\mathcal{I}_J}$ is positive.*

Proof. It can be proved as in [12], Proposition 4.5 that $\Delta_{\mathcal{I}}$ is not contained in a half-space. The result follows by theorem 7.1. □

If $\mathcal{I}_J = \emptyset$ then the previous proposition ensures that $\Delta = \Delta_\emptyset$ is not contained in a half-space. On the other hand, if $\mathcal{I}_J \neq \emptyset$ the fan Δ might still be contained in a half-space. We give next an algorithm to check whether Δ is contained in a half-space using the output set of algorithm 7.6. The algorithm is based on the following fact.

Lemma 7.8. *Let $\{\mathcal{I}_{l+1},\dots,\mathcal{I}_J\}$ be the output of* POSITIVITY-SET*(L_1, d). Given $k \in \{1,\dots,J-(l+1)\}$, if $\mathcal{I}_{J-k}=\emptyset$ and $\mathcal{I}_{J-k+1}\neq\emptyset$ then $\mathcal{C}=\mathbb{R}_{\leq 0}\{v_1,\dots,v_{J-k+1}\}$ is contained in $H_{v_i^*}$ for all $i\in\mathcal{I}_{J-k+1}$.*

Proof. Given $i\in\mathcal{I}_{J-j+1}$ then $i\in(\cap_{j=l+1}^{J-k}I_j^0)\cap I_{J-k+1}^+$. Thus by (22), (23), (24) and 4.1 in algorithm 7.6 we have $\langle v_i^*, v_j\rangle \geq 0$ for all $j=l+1,\dots,J-k+1$. This proves the result. □

Algorithm 7.9. HSP2*($\{\mathcal{I}_{l+1},\dots,\mathcal{I}_J\}$, $\mathcal{V}$)* Given *$\{\mathcal{I}_{l+1},\dots,\mathcal{I}_J\}$ the output set of algorithm 7.6 and $\mathcal{V}=\{v_1,\dots,v_r\}$ a set of vectors associated to the action of H on X, the algorithm* decides *whether $\mathcal{C}=\mathbb{R}_{\geq 0}\mathcal{V}$ is contained in a half-space and if the answer is affirmative it* returns *a vector $0\neq u\in M_{\mathbb{R}}$ such that $\mathcal{C}$ is contained in H_u.*

1. *$\mathcal{C}:=\mathbb{R}_{\geq 0}\{v_1,\dots,v_l\}$.*
2. *If $\mathcal{I}_{l+1},\dots,\mathcal{I}_J$ are nonempty sets then*
   - 2.1 *$\mathcal{C}:=\mathbb{R}_{\geq 0}(\mathcal{C}\cup\{v_{l+1}\})$, $u:=v_i^*$ with $i\in\mathcal{I}_{l+1}$.*
   - 2.2 *If $J=l+1$ return u and* "Contained in a half-space" *else return* HSP1*($\{v_{l+2},\dots,v_J\}$, $\mathcal{C}$, u).*
3. *If $\mathcal{I}_J=\emptyset$ then return* " Not contained in a half-space".
4. *Let $1\leq i\leq J-(l+1)$ be the smallest such $\mathcal{I}_{J-i}=\emptyset$ and $\mathcal{I}_{J-i+1}\neq\emptyset$.*
   - 4.1 *$\mathcal{C}:=\mathbb{R}_{\geq 0}(\mathcal{C}\cup\{v_{l+1},\dots,v_{J-i+1}\})$ and $u:=v_i^*$ with $i\in\mathcal{I}_{J-i+1}$.*
   - 4.2 *If $J-i+1=J$ return u and* " Contained in a half-space".
   - 4.3 *Return* HSP1*($\{v_{J-i+2},\dots,v_J\}$, $\mathcal{C}$, u).*

From the previous results we derive the following algorithm to test whether an A-grading is positive.

Algorithm 7.10. POSITIVITY-TEST*(L)* Given *the $m\times n$ matrix L associated to the action of $H=G\times\mathbb{F}$ on $X=k^r\times(k^\times)^s$ where G is a torus of dimension p, the algorithm* decides *whether the A-grading of S by L is positive and if the answer is negative it* returns *$\mathcal{I}\subset\{1,\dots l\}$ such that A-grading of S by $L_{\mathcal{I}}$ is positive.*

1. *If $p>s$ or $\eta_{r+1},\dots,\eta_n$ are linearly dependent return* "The grading is not positive".
2. $\begin{bmatrix} L_1 & dI_p \\ L_3 & L_4 \end{bmatrix}$:=SPECIAL-MATRIX*(L,D).*
3. *If any of the columns of L_1 is 0 then return* "The grading is not positive".
4. *$\{\mathcal{I}_{l+1},\dots,\mathcal{I}_J\}$:=*POSITIVITY-SET*(L_1,d).*
5. *If* HSP2*($\mathcal{I}_{l+1},\dots,\mathcal{I}_J$) returns* "Not contained in a half-space" *then return* "The grading is positive" *else return $\mathcal{I}_J$ and* "The grading is not positive".

8 Polyhedral description of graded components of S

In this section we describe the graded components of S in terms of polyhedra. We distinguish to main cases depending on the dimension.

Let Δ be a fan compatible with the multigrading of S by A given by L. Given $a \in Q$, there exists $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{N}^r \times \mathbb{Z}^s$ such that $L\varphi = a$. Consider the polyhedron $\mathcal{P}_\varphi$ defined by (3).

Lemma 8.1. *Let $a \in Q$ then for any $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{N}^r \times \mathbb{Z}^s$ such that $L\varphi = a$ it holds*

$$S_a = x^\varphi k[\omega(\mathcal{P}_\varphi \cap \mathbb{Z}^l)], \tag{26}$$

with ω as in (16).

Proof. For each $a \in Q$ we have $S_a = k[(\varphi + \mathfrak{K}) \cap (\mathbb{N}^r \times \mathbb{Z}^s)]$. Given $\sigma \in \Delta$ let

$$\phi_{\sigma,a} = \{\varphi + \mu \in \varphi + \mathfrak{K} \mid (\mu, e_i) \geq -\varphi_i \text{ for all } i \in [\sigma]\}. \tag{27}$$

Then $S_a = k[\cap_{\sigma\in\Delta}\phi_{\sigma,a}]$. Define the sets

$$\psi_{\sigma,a} = \{\lambda \in \mathfrak{K} \mid (\lambda, e_i) \geq -\varphi_i, \text{ for all } i \in [\sigma]\}. \tag{28}$$

Thus $\phi_{\sigma,a} = \varphi + \psi_{\sigma,a}$ and $k[\phi_{\sigma,a}] = x^\varphi k[\psi_{\sigma,a}]$. Therefore $S_a = x^\varphi k[\cap_\sigma \psi_{\sigma,a}]$. Let

$$\Lambda_{\sigma,a} = \{x \in M \mid \langle x, v_i\rangle \geq -\varphi_i \text{ for all } i \in [\sigma]\}. \tag{29}$$

Then $\psi_{\sigma,a} = \omega(\Lambda_{\sigma,a})$ and $\omega(\cap_{\sigma\in\Delta}\Lambda_{\sigma,a}) = \cap_\sigma \psi_{\sigma,a}$. Finally $\cap_{\sigma\in\Delta}\Lambda_{\sigma,a} = \mathcal{P}_\varphi \cap \mathbb{Z}^l$. □

The following theorem gives two more properties of positive gradings.

Theorem 8.2. *The following statements are equivalent:*

1. *There exits $a \in A$ such that the k-vector space S_a is finite dimensional.*
2. *For all $a \in A$, the k-vector space S_a is finite dimensional.*
3. *Any fan compatible with the multigrading of S given by A is not contained in a half-space.*

Proof. Let Δ be a fan compatible with the multigrading of S by A. Given $\varphi \in \mathbb{N}^r \times \mathbb{Z}^s$, by proposition 3.3 the set $\mathcal{P}_\varphi \cap \mathbb{Z}^l$ is finite if and only if Δ is not contained in a half-space. Given $a \in Q$ and $\varphi \in \mathbb{N}^r \times \mathbb{Z}^s$ such that $L\varphi = a$ by the previous lemma S_a is finite dimensional if and only if $\mathcal{P}_\varphi \cap \mathbb{Z}^l$ is finite. This proves the result. □

8.1 Finite dimensional case

At this point we can determine the graded components in the finite dimensional case.

Theorem 8.3. *Let us suppose that the grading of S by A given by L is positive. Given $a \in Q$, then the dimension of S_a equals the number of lattice points inside the polytope $\mathcal{P}_\varphi$ for any φ such that $L\varphi = a$.*

Proof. By proposition 3.3, the polyhedron $\mathcal{P}_\varphi$ is bounded. By lemma 8.1 the graded component S_a is spanned over k by the finite set of monomials x^u such that

$$u \in \varphi + \omega(\mathcal{P}_\varphi \cap \mathbb{Z}^l).$$

This proves the result □

Remark 8.4. POLYHEDRAL DESCRIPTION *The proof of theorem 8.3 provides an algorithm to determine S_a using a polytope.*

Let us suppose that the grading of S by A given by L is positive, that is $S_0 = k$. Then by [10], Lemma 4.1(1) the vectors $\eta_{r+1}, \ldots, \eta_n$ are linearly independent and we can assume that L is of the special kind (14). Let $\{v_1, \ldots, v_r\}$ be a set of vectors associated to the action of H on X. By lemma 7.4, the set $\bar{\mathcal{B}} = \{-v_1 \ldots, -v_l\}$ is an $N_\mathbb{R}$ basis. Let us consider the $r \times l$ matrix P whose i-th row is the row vector of the coordinates of v_i in the $N_\mathbb{R}$ basis $\bar{\mathcal{B}}$, $i = 1 \ldots r$. In the dual basis of $\bar{\mathcal{B}}$ the polytope $\mathcal{P}_\varphi$ equals the set

$$\{x \in \mathbb{R}^l | Px \leq \varphi_i, i = 1, \ldots, r\}. \tag{30}$$

8.2 Infinite dimensional case

We describe next the graded components of S for the infinite dimensional case. Let us suppose that S_0 is not finite dimensional. Let $\mathcal{V} = \{v_1, \ldots, v_r\}$ be a set of vectors associated to the action of H on X and let Δ be a fan compatible with the multigrading of S by A. By theorem 7.1, Δ is contained in a half-space and by remark 3.4, the set $\mathcal{C} = \mathbb{R}_{\geq 0}\{v_1, \ldots, v_r\}$ is a cone. Let $\mathcal{C}^\vee$ be the dual cone of $\mathcal{C}$. By [13], Theorem 16.7 there exists a Hilbert basis of $\mathcal{C}^\vee$.

Proposition 8.5. *Let S_0 be infinite dimensional and let $\mathcal{H} = \{w_1, \ldots, w_h\}$ be a Hilbert basis of $\mathcal{C}^\vee$. Then*

$$S_0 = k[x^{\omega(w_1)}, \ldots, x^{\omega(w_h)}].$$

Proof. For each $\sigma \in \Delta$, let ϕ_σ be defined as in (17) and Λ_σ as in eq1. Then $\cap_{\sigma\in\Delta}\phi_\sigma = \mathfrak{K} \cap (\mathbb{N}^r \times \mathbb{Z}^s)$, $\cap_{\sigma\in\Delta}\Delta_\sigma = \mathcal{C}^\vee \cap M$ and $\omega(\cap_{\sigma\in\Delta}\Delta_\sigma) = \cap_{\sigma\in\Delta}\phi_\sigma$. Thus $\{\omega(w_1), \ldots, \omega(w_h)\}$ is a minimal generating set of the partially ordered set $\mathfrak{K} \cap (\mathbb{N}^r \times \mathbb{Z}^s)$ and this proves the result. □

Every graded component of S is infinite dimensional. Given $a \in Q$ let $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathbb{N}^r \times \mathbb{Z}^s$ such that $L\varphi = a$. The next result will be used to determine S_a.

By proposition 3.3, the polyhedron $\mathcal{P}_\varphi$ is not bounded. By [13], Corollary 7.1b then $\mathcal{P}_\varphi = \mathcal{P} + \mathcal{C}^\vee$ for some polytope $\mathcal{P}$.

Lemma 8.6. *Let $\mathcal{H} = \{w_1, \dots, w_h\}$ be a Hilbert basis of $\mathcal{C}^\vee$, let $B = \{\sum_{i=1}^h \alpha_i w_i \mid 0 \le \alpha_i \le 1\}$ and let $\mathcal{P}$ be a polytope such that $\mathcal{P}_\varphi = \mathcal{P} + \mathcal{C}^\vee$. Then*

$$\mathcal{P}_\varphi \cap \mathbb{Z}^l = ((\mathcal{P} + B) \cap \mathbb{Z}^l) + (\mathcal{C}^\vee \cap \mathbb{Z}^l) \tag{31}$$

Proof. We prove the nontrivial inclusion $\mathcal{P}_\varphi \cap \mathbb{Z}^l \subseteq ((\mathcal{P}+B)\cap\mathbb{Z}^l) + (\mathcal{C}^\vee \cap \mathbb{Z}^l)$. Given $p \in \mathcal{P}_\varphi \cap \mathbb{Z}^l$ then $p = q + c$ with $q \in Q$ and $c \in \mathcal{C}^\vee$. We can write $c = \sum \alpha_i w_i$ with $\alpha_i \ge 0$. Consider the integral vector $c' = \sum \lfloor \alpha_i \rfloor w_i$, thus $b = c - c' \in B$. We can write $p = q + b + c'$ with $q + b = p - c' \in (\mathcal{P} + B) \cap \mathbb{Z}^l$ and $c' \in \mathcal{C}^\vee \cap \mathbb{Z}^l$. □

Given a set of generators of a cone, we can use the algorithm given in [6], section 5.5 to compute a Hilbert basis of the dual cone.

Theorem 8.7. *For all $a \in Q$, S_a is finitely generated as an S_0-module.*

Proof. Let us consider the cone $\bar{\mathcal{C}}$ generated by vectors $(v_i, \varphi_i) \in \mathbb{Z}^{l+1}$, $i = 1, \dots, r$ and $(0_{\mathbb{Z}^l}, 1) \in \mathbb{Z}^{l+1}$ with $0_{\mathbb{Z}^l}$ the zero vector in $\mathbb{Z}^l$. By [13], Theorem 16.7 there exists a Hilbert basis $\mathcal{H}$ of the dual cone $\bar{\mathcal{C}}^\vee$ of $\bar{\mathcal{C}}$ which equals

$$\{(x, \lambda) \in \mathbb{R}^l \times \mathbb{R}_{\ge 0} \mid \langle x, -v_i \rangle - \lambda\varphi_i \le 0\}$$

The set $\mathcal{H}_0 = \{(x, \lambda) \in \mathcal{H} \mid \lambda = 0\}$ is a Hilbert basis of $\mathcal{C}^\vee$. Let $\mathcal{P}$ be the convex hull of the $x \in \mathbb{Z}^l$ with $(x, 1) \in \mathcal{H}$. Now $x \in \mathcal{P}_\varphi$ if and only if $(x, 1) \in \bar{\mathcal{C}}^\vee$. Then $\mathcal{P}_\varphi = \mathcal{P} + \mathcal{C}^\vee$.

By lemma 8.1 and lemma 8.6 we conclude that S_a is generated as an S_0-module by the elements x^u such that $u \in \varphi + \omega((\mathcal{P} + B) \cap \mathbb{Z}^l)$. □

Remark 8.8. POLYHEDRAL DESCRIPTION *The proof of theorem 8.7 provides an algorithm to determine S_a using polyhedra.*

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